ON EMBEDDING A GRAPH IN THE PLANE

BY

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DEPARTMENT OF ELECTRICAL ENGINEERING
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CERTIFICATE

Certified that this work on " ON EMBEDDING A GRAPH IN THE PLANE" by Mr. V.G.Kane has been carried out under my supervision and that this has not been submitted elsewhere for a degree.

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V.G.Kane

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SYNOPSIS

One of the important problems in Graph Theory is to represent a non-planar graph in the plane with minimum number of intersections. This problem has not been solved in general. While considering this problem, we have defined the concept of the depth of a planar graph. It is proved that for every non-negative integer s, there exists a planar graph of depth s. An algorithm is also given to estimate the number of intersections of a particular kind in a representation of a graph with vertices either on the x-axis or on the y-axis.

CHAPTER I

Let G be a finite graph. A graph G is said to be planar if it can be embedded on a plane in such a fashion that the vertices are all distinct points, the edges are simple curves, and no two edges meet one another except at their terminals. A diagram of G, on a plane, which conforms with these conditions is called a planar representation of G. Two planar representations will be regarded as distinct if they cannot be made to coincide with one another by elastic deformation of the plane (6). Any simple closed curve in a plane divides the plane into two areas (Jordan's theorem). A circuit of G is a simple closed curve in any planar representation of G. A region of a planar representation of a graph G is an area of the plane determined by a circuit of G, which does not contain an area determined by any other circuit of G. The circuit of G, which determines a region in a planar representation of G, is called the boundary of the region and the edges in the circuit are referred to as the edges forming the region. A region with a finite (infinite) area is called a finite (infinite) region. A planar representation of a graph G has exactly one infinite region, which will be called the exterior regic A planar graph can be embedded onto a plane in such a way that any chosen region will become the exterior region ((1), pp 209).

Let G be a finite, connected, planar graph with p vertices and q edges. Let A be the set of all distinct planar representations of G. Our aim is to show that the set A is a finite set.

First we will prove that for a graph G with $q = p-1$ the set A is a finite set. In this case G is a tree. If $p = 1$, then clearl the set A is a finite set. Assume that, every tree with n-1 vertices has finitely many distinct planar representations. Let G be a tree with n vertices. G has at least two pendent vertices. Remove the edge incident on some pendent vertex of G. Call this graph G^1. G^1 has two components, G^1_1 (a tree with n-1 vertices) and G^1_2 (an isolated vertex). By induction hypothesis, G^1_1 has finitely many distict planar representations. Hence G has finitely many distinct planar representations. Hence the set A is a finite set.

Assume that all graphs with p vertices and e edges $(e>p-1)$ have finitely many distict planar representations. Let G have p vertices and e+1 edges. Clearly G has a circuit. Remove an edge from G which is in some circuit of G. Thus we get a connected subgraph G^1 of G with p vertices and e edges. If G^1 is a tree, then G^1 has finitely many distinct planar representations. If G^1 is not a tree, then by induction hypothesis G^1 will have finitely many distinct planar representations. Hence G also has finitely many distinct planar representations.

Hence the set A is a finite set. A graph G with p vertices and q edges is called a (p,q) graph.

Definition 1: Let G be a connected (p,q) graph. Let $v \varepsilon V$, $T \varepsilon A$. Define the depth of v in T as the minimum number of edges that have to be crossed to reach the vertex v from the exterior region of G in T. Denote the depth of v in T by $\mathrm{Dep}_T(v)$.

Definition 2: Let G be a connected (p,q) graph. Define the depth of G in T, $T \varepsilon A$, as follows and denote it by $\mathrm{Dep}_T(G)$.

$$\mathrm{Dep}_T(G) = \max_{v \varepsilon V} (\mathrm{Dep}_T(v))$$

Definition 3: Finally, we introduce the concept of the depth of a connected (p,q) graph G as follows and denote it by Dep (G).

$$\begin{aligned} \mathrm{Dep}(G) &= \min_{T \varepsilon A} (\mathrm{Dep}_T(G)) \\ &= \min_{T \varepsilon A} \max_{v \varepsilon V} \mathrm{Dep}_T(v) \end{aligned}$$

Definition 4: Let G be a disconnected (p,q) graph. Let $G_1, G_2, \ldots\ldots, G_s$ be the connected components of G . Define the depth of G as follows:

$$\mathrm{Dep}(G) = \max_i \mathrm{Dep}(G_i), \quad i = 1,2,\ldots.,s$$

Let the depth of G be n. This implies that for every $T \varepsilon A$, there exists at least one vertex $v \varepsilon V$, such that, to reach this vertex v from the exterior region of G in T, at least n edges have to be crossed; and, furthermore, there exists $\overset{*}{T'} \varepsilon A$,in which, every $v \varepsilon V$ can be reached from the exterior region by crossing at most n edges.

Unless otherwise stated, by G, we will always mean a finite simple planar graph. The concept of an outerplanar graph was first defined by Chartrand and Harary in 1967.(7) This concept will be useful in our further discussion. We give below the definition of an outerplanar graph and state some known results about them, given in (2), (See pp 106-107 in (2)).

Definition 5: A planar graph G is called outerplanar if it can be embedded in the plane so that all the vertices lie on the exterior region.

Definition 6: An outerplanar graph G is called maximal outerplanar if no edge can be added to it without loosing outerplanarity.

A maximal outerplanar graph is a triangulation of a polygon.

The following theorem and corollary are known about maximal outerplanar graphs.

Theorem 1 :

Let G be a maximal outerplanar (p,q) graph with $p \geq 3$. Then G has p-2 regions (excluding the exterior region).

Corollary 1.1:

Every maximal outerplanar (p,q) graph G has

a) $q = 2p-3$ edges,

b) at least three vertices of degree not exceeding 3,

c) at least two vertices of degree 2 .

Remark: The following are the immediate consequences of the definition of the depth of a (p,q) graph G and the above results, about maximal outerplanar graphs.

A) A graph G is outerplanar iff Dep(G)=0 (By definition 5 and definition 4) .

B) Let G be a (p,q) graph. If $q > 2p-3$, then $\text{Dep}(G) \geq 1$ (By corollary 1.1 (a)) .

C) If the degree of each vertex of G is at least 3, then $\text{Dep}(G) \geq 1$ (By Corollary 1.1 (c)).

D) If the degree of each vertex is odd and if $q=2p-3$, then $\text{Dep}(G) \geq 1$ (By Corollary 1.1 (a) and Corollary 1.1 (c)) .

E) If G has no vertex of degree 2 or at most one vertex of degree 2 and if $q=2p-3$, then $\text{Dep}(G) \geq 1$ (By Corollary 1.1 (a) and Corollary 1.1 (c)) .

F) If G' is a subgraph of G, then $\text{Dep}(G') \leq \text{Dep}(G)$.

Now we proceed to generalise the notion of an outerplanar graph.

Definition 7: Let G be a (p,q) graph, with $p \geq 3$. Then G is called k-outerplanar $(3 \leq k \leq p)$ if it can be embedded in the plane so that k vertices lie on the exterior region and it cannot be embedded in the plane so that more than k vertices lie on the exterior region.

Definition 8: A k-outerplanar graph G is called maximal k-outerplanar if no edge can be added to G without loosing k-outerplanarity.

Let G be a (p,q) graph with $p \geq 3$. Let G be maximal 3-outerplanar. Obviously, it is a maximal planar graph and therefore it has 3p-6 edges.

Since the concepts of outerplanarity and p-outerplanarity for a (p,q) planar graph G are equivalent, clearly, a maximal p-outerplanar graph has 2p-3 edges.

Lemma 1:

Let G be a (p,q) graph. Let $N = 3p-6-q$. Then any region of G can have at most N+3 edges.

Proof:

Case 1: G is connected.

Let, if possible, G have a region with N+3+t edges, where $t > 0$. Since G is a connected (p,q) graph, G has $r=q-p+2$ regions, including the exterior region, (Euler Polyhedron Formula, see (2), pp 103). From our assumption, G has at least one region with N+3+t edges. Now, each of the remaining regions has at least 3 edges.

Therefore, we have,

$$q \geq \frac{(N+3+t) + (q-p+1)3}{2}$$

but $q = 3p-6-N$

$$\therefore\ 2(3p-6-N) \geq N+3+t+(3p-6-N-p+1)3$$

$$\therefore\ 6p-12-2N \geq N+3+t+9p-18-3N-3p+3$$

$$\therefore\ t \leq 0$$

This contradicts the assumption $t > 0$. Hence, any region of G has at most N+3 edges.

Case 2: G is disconnected.

Let $G_1, G_2, \ldots\ldots, G_s$ be the connected components of G, $s \geq 1$. Let G_i be a (p_i, q_i) graph, for $i=1,2,\ldots,s$.

$$\sum_{i=1}^{s} p_i = p \quad , \quad \sum_{i=1}^{s} q_i = q$$

Let $N_i = 3p_i - q_i - 6 \qquad i = 1,2,\ldots\ldots,s.$

We have proved that Lemma 1 is true for all connected graphs G. Therefore, it is true for each of these components. Therefore, any region of G_i can have at most N_i+3 edges, for $i = 1,2,\ldots,s$.

Let M denote the maximum number of edges a region in G can have. Then,

$$M \leq \sum_{i=1}^{s} (N_i+3)$$

$$= \sum_{i=1}^{s} (3p_i - q_i - 6+3)$$

$$= 3\sum_{i=1}^{s} p_i - \sum_{i=1}^{s} q_i - 3s$$

$$= 3p-q-3s$$

$$= N+6-3s$$

Since $s \geq 1$

$M \leq N+3$

which completes the proof of Lemma 1.

Lemma 2:

If G is a maximal k-outerplanar (p,q) graph, then $q = 3p-k-3$.

Proof:

In a maximal k-outerplanar graph, there is one region with k edges and other regions have 3 edges each. Hence,

$$q = \frac{k+3(q-p+1)}{2}$$

$$\therefore \quad 2q = k+3q-3p+3$$

$$\therefore \quad q = 3p-k-3$$

This completes the proof of Lemma 2.

Remark:

In Lemma 2, if we put $k=3$ and $k=p$ we get the number of edges in the cases of maximal planar and maximal outerplanar graphs, respectively.

Lemma 3:

Let G be a (p,q) graph. Let R be any region of G, formed by maximum number of edges equal to e. If $q-e > 2p-3$, then $\text{Dep}(G) \geq 2$.

Proof:

Let $T \in A$. Let R_T be the exterior region of G in T. Let R_T have e_T edges. Remove these e_T edges from G to get a subgraph G_T of G. G_T is a $(p, q-e_T)$ graph.

Since $e_T \leq e$, we have,

$$q - e_T \geq q - e$$

but $$q-e > 2p-3$$

$$\therefore \quad q - e_T > 2p-3$$

$$\therefore \quad Dep(G_T) \geq 1$$

Now, G_T is enclosed in the circuit formed by the edges (which form the region R_T)

$$\therefore \quad Dep_T(G) \geq 2$$

Since T is an arbitrary element of A, we have,

$$Dep(G) = \min_{T \in A} Dep_T(G) \geq 2.$$

This completes the proof of Lemma 3.

Remark:

For every $T \in A$, we get a subgraph G_T of G. If all these subgraphs satisfy the conditions of Lemma 3, then $Dep(G_T) \geq 2$ for all T. Hence $Dep(G) \geq 3$. In other words, if $Dep(G_T) \geq m$ for every $T \in A$, then

$$Dep_T(G) \geq m+1$$

$$\therefore \quad Dep(G) = \min_{T \in A} Dep_T(G) \geq m+1$$

We can try to determine the value of m, as far as possible by the application of Lemma 3 to these subgraphs, till any one of the intermediate subgraphs fails to satisfy the conditions of Lemma 3. It is not always possible to find this number easily, but an upper bound can be easily determined which is $u = 3p-q-3$. At each stage of determining m we can check, whether $q-u > 2p-3$ or not.

The following characterisation of outerplanar graphs was given by Chartrand and Harary (See (2), pp 107).

A graph is outerplanar iff it has no subgraphs homeomorphic to K_4 or $K_{(2,3)}$, except $K_4 - x$.

Thus, in some sense, these two subgraphs are the basic non-outerplanar graphs. Obviously $\mathrm{Dep}(K_4) \geq 1$, $\mathrm{Dep}(K_{(2,3)}) \geq 1$. Now we give representations of these graphs where every vertex can be reached from the exterior region by crossing at most 1 edge. Thus $\mathrm{Dep}(K_4) = 1$ and also $\mathrm{Dep}(K_{(2,3)}) = 1$. See figure 1.

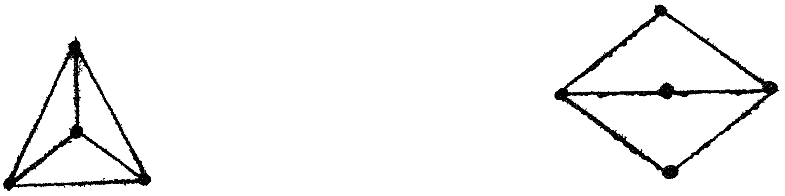


Figure 1

Theorem 2:

Let G be a (p,q) graph. If $q > 3(p-1) - \dfrac{p}{2^{s-1}}$, then $\mathrm{Dep}(G) \geq s$.

Proof:

The theorem is true for $s=1$ (By Remark (B) on page 4). Let the theorem be true for all (p,q) graphs G with

$q > 3(p-1) - \frac{p}{2^{s-1}}$. Let G be a (p,q) graph with $q > 3(p-1) - \frac{p}{2^{s}}$. A region in G can have at most $r^* = 3p-q-3$ edges (by Lemma 1).

$$\therefore\ q-r^* = q-(3p-q-3)$$

$$= -3p+2q+3$$

but $$q > 3(p-1) - \frac{p}{2^{s}}$$

$$\therefore\ q-r^* > -3(p-1)+6(p-1) - \frac{p}{2^{s-1}}$$

$$= 3(p-1) - \frac{p}{2^{s-1}}$$

Now let $T \varepsilon A$ and let R_T be the exterior region of G in T. Let G_T be the subgraph of G obtained by removing all the edges forming the region R_T. Let G_T be a (p,q_T) graph.

Now $$q_T \geq q-r^* > 3(p-1) - \frac{p}{2^{s-1}}$$

Therefore, by hypothesis,

$$\text{Dep}\ (G_T) \geq s$$

$$\therefore\ \text{Dep}_T(G) \geq s+1$$

$$\therefore\ \text{Dep}\ (G) = \min_{T \varepsilon A} \text{Dep}_T(G)$$

$$\geq s+1$$

Hence the theorem follows.

Corollary 2.1:

Let G be a maximal k-outerplanar (p,q) graph. If $k < \frac{p}{2^{s-1}}$, then $\text{Dep}\ (G) \geq s$.

Proof:

G has $q = 3p-k-3$ edges (by Lemma 2).

But $k < \frac{p}{2^{s-1}}$

$$\therefore \quad q > 3(p-1) - \frac{p}{2^{s-1}}$$

Hence by theorem 2, Dep $(G) \geq s$.

Corollary 2.2:

If G is a $(p, 3p-6)$ graph, $3\ 2^s \geq p > 3\ 2^{s-1}$, then Dep. $(G) \geq s$.

Proof:

Since G is maximal 3-outerplanar, $k=3$.

Since $p > 3\ 2^{s-1}$

$$3 < \frac{p}{2^{s-1}}$$

$$\therefore \quad 3 = k < \frac{p}{2^{s-1}}$$

$\therefore$ By corollary 2.1, Dep $(G) \geq s$.

Now we ask the following question:

Does there exist, a graph of depth s for every non-negative integer s ? The following theorem answers the question, affirmatively.

Theorem 3:

For every non-negative integer s, there exists a graph of depth s.

Proof:

We prove the above theorem by constructing a graph of depth s, for every non-negative integer s.

Let G be a $(p,3p-6)$ graph , where $p=3\ 2^{s}$, for some integer s. Since G is a maximal planar graph, by corollary 2.2, $\mathrm{Dep}(G) \geq s$.

Now we proceed to construct a maximal planar graph G with $3\ 2^{s}$ vertices, whose depth is exactly equal to s.

Construction:

Draw a polygon with $3\ 2^{s}$ vertices. Along the polygon, label the vertices as $a_1, a_2, \ldots, a_p$, where $p= 3\ 2^{s}$. Triangulate the polygon by joining a_1 to a_3 , a_4, ...,a_{p-1}. Let this graph be denoted by G^0.

G^0 is maximal $3\ 2^{s}$-outerplanar.

$\therefore$ $\mathrm{Dep}\ (G^0) = 0$.

Join a_2 to a_4,a_4 to a_6, ..., a_{p-2} to a_p,a_p to a_2. Thus,we get a maximal $3\ 2^{s-1}$-outerplanar graph.Call this graph G^1. Here $k= 3\ 2^{s-1}$ and $k < \frac{p}{2^{1-1}}$. Therefore, by corollary 2.1,$\mathrm{Dep}(G^1) \geq 1$. If we remove the exterior region of G^1, we get the graph G^0.
Hence, $\mathrm{Dep}(G^1) \leq \mathrm{Dep}(G^0)+1 =1$. Therefore, $\mathrm{Dep}(G^1)=1$.

Construct G^2 from G^1 by joining a_2 to a_6 , a_6 to a_{10},....., a_{p-2} to a_2 . Thus we get a maximal $3\ 2^{s-2}$-outerplanar graph. By a similar argument, we can show that $\mathrm{Dep}(G^2) = 2$. If we continue this process of joining the vertices,on the exterior region, alternately , finally we get a graph G^s, which is maximal 3-outerplanar (**maximal** planar) and it is clear that $\mathrm{Dep}(G^s) = s$.

This completes the proof of the theorem.

Note that in this construction, we get graphs with depths equal to $0,1,2,\ldots\ldots,s-1$ as intermediate subgraphs.

Lemma 4:

The (12,30) graph G shown in figure 2, which is regular of degree 5, has the depth 3.

Proof:

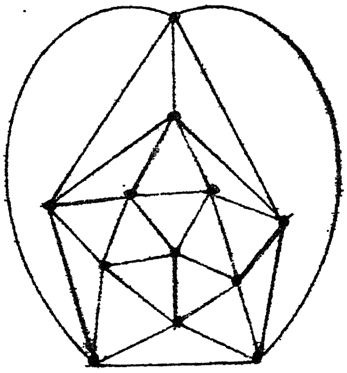


Figure 2

The graph G, shown in the above figure, is the one under consideration. The degree sequence of G is ,

5,5,5,5,5,5,5,5,5,5,5,5

Since $12 > 3\ 2^{s-1}$ for $s = 2$, by corollary 2.2, $\mathrm{Dep}\ (G) \geq 2$.

Let $T \epsilon A$. Let R_T be the exterior region of G in T and G_T be the corresponding subgraph of G obtained by removing all the edges forming R_T. Since every region of G has 3 edges, so G_T has 27 edges, for every $T \varepsilon A$. If we can show that Dep $(G_T) \geq 2$ for every $T \varepsilon A$, then clearly Dep(G) ≥ 3 and since in the representation of G in the figure 2, every vertex can be reached by crossing at most 3 edges from the exterior region, it follows that Dep(G) = 3.

Claim: Dep$(G_T) \geq 2$ for every $T \varepsilon A$.

G_T has 12 vertices and 27 edges. By corollary 1.1(a), G_T is non-outerplanar and Dep$(G_T) \geq 1$. The degree sequence of G_T is

$$5,5,5,5,5,5,5,5,5,3,3,3$$

A region in G_T can have at most 30-27+3=6 edges (by Lemma 1).

$$G_T \text{ has } r_T = 27-12+2$$
$$= 17 \text{ regions.}$$

We had 20 regions in G. After removing edges forming a region R_T, 4 regions were disturbed (since every edge is contained in exactly two regions) and 16 regions were undisturbed. Therefore, in G_T we have 16 regions, each formed by 3 edges. Let the remaining region be formed by e edges.

$$\therefore \quad 27 = \frac{e+16\times 3}{2}$$

$$\therefore \quad e = 6$$

Obviously, the three vertices of degree 3 in G_T must be on the boundary of the region with 6 edges. Therefore, after removing a region formed by 6 edges, we get, a subgraph of G_T, say G_T' which has 21 (the number $2p-3$, for $p=12$) edges and its degree sequence is ,

5,5,5,5,5,5,3,3,3,1,1,1

G_T' has 21 edges but does not have a vertex of degree 2 at all . Hence by Remark E on page 4, we get,

G_T' is non-outerplanar.

$\therefore$ Dep$(G_T') \geq 2$, for every $T \in A$

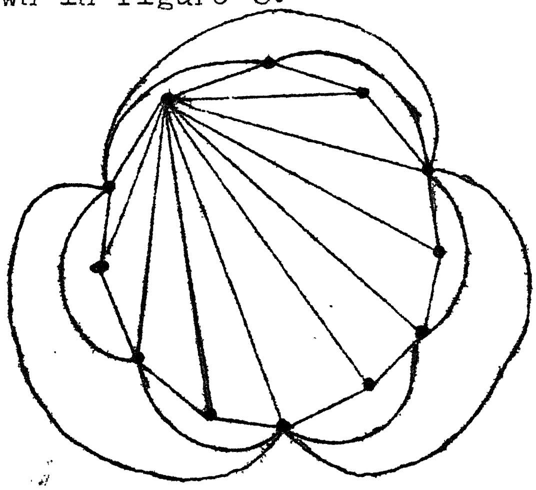
$\therefore$ Dep$(G) = 3$

Hence the Lemma is proved.

Remark:

We now observe an interesting fact, viz. two maximal planar graphs with same number of vertices need not have the same depth.

In Lemma 4, we have a 12-vertex maximal planar graph whose depth is 3. Now we construct another 12-vertex maximal planar graph whose depth is 2. The procedure will be the same as described in the proof of Theorem 2. This graph is shown in figure 3.



(A maximal planar graph with 12 vertices of depth 2)

Figure 3

Method of determining the depth of a graph:

Here let G denote any finite planar graph (not necessarily simple). Let D_G denote the set of all non-isomorphic duals of G. D_G is a finite set, since the number of circuits of G are finite.

Let $G^* \varepsilon D_G$.

Let $V = \{v_1, v_2, \ldots v_p\}$ be the vertex set of G.

Let R^* denote the vertex set of G^* (This is also the set of all regions of G, in some planar representation of G).

Let $R_{vi} = \{R \varepsilon R^* | v_i$ lies on the boundary of $R\}$. Let $R, S \;\varepsilon\; R^*$.

Let $d(R,S)$ denote the length of a shortest path between R and S in G^*.

Define

$$\Delta = \min_{G^* \varepsilon D_G} \quad \min_{R \varepsilon R^*} \quad \max_{v_i \varepsilon V} \quad \min_{S \varepsilon R_{vi}} \quad d(R,S)$$

Lemma 5:

Δ is the depth of G.

Proof:

Let

$$K_{R,v_i} = \min_{S \varepsilon R_{vi}} \quad d(R,S)$$

Here we are fixing v_i and R. By the definition of R_{v_i}, v_i lies on the boundary of all regions which belong to R_{v_i}. Determine K_{R,v_i}. Then K_{R,v_i} gives the minimum number of edges that have to be crossed to reach the vertex v_i from the region R. Now, let $K_R = \max_{v_i \in V} K_{R,v_i}$. Then K_R gives the maximum number of edges that have to be crossed to reach any vertex of G from R. Now it is obvious that minimum of $\min_{R \in R^*} K_R$, over all $G^* \in D_G$, will be the depth of G. Hence the Lemma.

Remark:

The region (or regions) for which $\max_{v_i \in V} \min_{S \in R_{v_i}} d(R,S) = \Delta$, is precisely the one which should be choosen as the exterior region to get the representation of G in which every vertex can be reached from R by crossing at most Δ edges.

Open Problem:

Let G^n denote the collection of all planar graphs with n vertices. Let us denote by d_n^* the largest depth that a planar graph with n vertices can have. Determine this number d_n^* for each non-negative integer n. This will then give the upper bound on the depth of a graph with n vertices, attained by some $G \in G^n$.

CHAPTER II

The non-planar graphs arise in practice in many different places . The main problem, here, is to draw a non-planar graph in the plane, with minimum number of intersections. This problem has not been solved in general; but the solutions, in terms of upper bounds have been obtained for some special classes of graphs. Zarankiewicz (see (3)) has considered this problem for complete bipartite graphs. T.A.J.Nicholson (see (4)) has described a procedure wherein the intersections obtained are minimum or near to the minimum intersections. He gives the representation of the graph G by placing the vertices of G on a straight line in the plane. Hashimoto (see (5)) has pointed out that Nicholson's procedure does not work for a planar graph shown in figure 4.. Nicholson represents all the vertices of G on a straight line and gives a procedure to obtain a 2-optimal permutation (that permutation in which interchanging of any two vertices will not reduce the number of intersections) of the vertices. The failure of Nicholson's algorithm suggests that it may not be advisable to have all the vertices on a straight line. We can have some vertices along the x-axis and the rest along the y-axis in the co-ordinate plane. The planar graph given in (5), (shown in figure (4)) for which the Nicholson's method fails, can in particular be represented in the above way. (see figure (5)). If

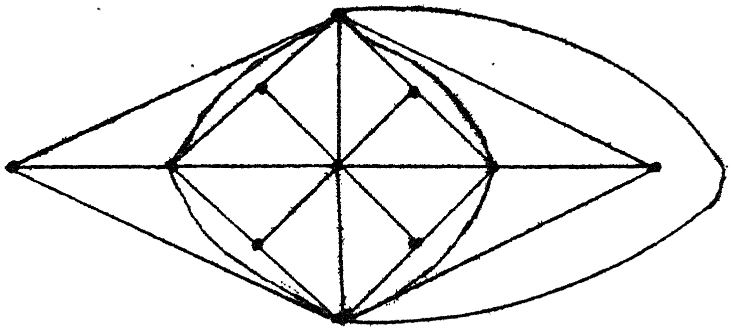


Figure 4

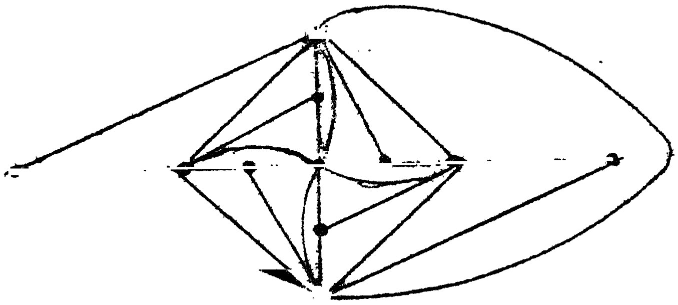


Figure 5

we want to represent the graph in the above fashion, the following problems arise:

1. The first problem is to give a criterian to decide which vertices should be placed along a particular axis.

2. The second one is to decide how the connections should be made between the vertices.

3. The third problem is to give a method to estimate the number intersections in a particular representation of G.

Some of the above problems are considered below:

Zarankiewicz gave a representation which realises the minimum number of intersections in complete bipartite graphs. He represents one set of vertices along x-axis and the rest along y-axis, such that, all the connections are from x-axis to y-axis. The connections are made by straight lines. We suggest that in the x-y axis representation of any graph G, the connections between vertices on x-axis to those on y-axis should be made by straight line edges. The connections between the vertices of x-axis (y-axis) will be made by parts of semi-circles. In this case we can use the formulae given by Nicholson to estimate the number of intersections. In what follows, a method is described to estimate the number of intersections, in a x-y axis representation, between the st. line edges.

Precisely, we want to know the number of intersections in each of the quadrants of the co-ordinate plane, between the straight line edges. The sum of the intersections, in all the four quadrants, will give the total number of intersections between the straight lines in a given representation.

Consider, in particular, the second quadrant. Let there be n vertices on the negative side of the x-axis and m vertices on the positive side of the y-axis. Let the connection matrix M be as shown below. Let the x-co-ordinates of the vertices on x-axis be $x_1, x_2, \ldots\ldots, x_n$ and the y-co-ordinates of the vertices on y-axis be $y_1, y_2, \ldots\ldots, y_m$. The $(i,j)^{th}$ entry of M will be 1 if there is an edge joining the vertices $(x_j, 0)$ and $(0, y_i)$; otherwise the $(i,j)^{th}$ entry will be zero.

$$M = \begin{array}{cccccc l}
1 & 0 & 0 & \cdots & 0 & 0 & y_1 \\
0 & 1 & 0 & \cdots & 1 & 1 & y_2 \\
 & \vdots & & \vdots & & \vdots & \\
1 & 0 & 1 & \cdots & 1 & 0 & y_m \\
x_1 & x_2 & x_3 & \cdots & & x_m &
\end{array}$$

Let $L^{1,k}$ be the line joining $(x_k,0)$ and $(0,y_1)$ and $L^{q,p}$ be the line joining $(x_p,0)$ and $(0,y_q)$. Now it is easy to see that $L^{1,k}$ and $L^{q,p}$ intersect if and only if

1. $|x_k| > |x_p|$ and $|y_1| < |y_q|$

or

2. $|x_k| < |x_p|$ and $|y_1| > |y_q|$

Let $(i,j)^{th}$ entry of M be 1. Let $L^{j,i}$ be the corresponding line joining $(x_j,0)$ and $(0,y_i)$. We can find out, by making use of the conditions stated above, with what lines, the line $L^{j,i}$ intersects. If, for each line, we find out the number of lines with which it intersects, then the addition of all these numbers will give twice the total number of intersections, since each intersection is counted twice. Suppose that we have fixed a particular line, say $L^{1,k}$. Then we find the number of lines which satisfy the condition 1, with the line $L^{1,k}$. This we do for each line $L^{1,k}$. The addition of all these numbers will then, give us the total number of intersections. Now we give a simple algorithm in which we construct two matrices from M. The sum of the elements, on the principal diagonal of the product of the above two matrices, which will be constructed, gives the total number of intersections.

Consider the line $L^{1,1}$. No line $L^{i,j}$ satisfies the condition 1 with $L^{1,1}$. In fact no line $L^{p,q}$ satisfies the condition 1 with $L^{r,1}$ (for $r = 1,2,\ldots,m$) and $L^{m,s}$ (for $s = 1,2,\ldots,n$). Consider the line $L^{j,i}$. It will satisfy the condition 1 with lines joining vertices $i+1, i+2,\ldots,m$ on y-axis and vertices $1,2,\ldots,j-1$ on x-axis.

Let us define two matrices U^* and L^* as follows:

Let U^* be the matrix obtained by deleting the first column and the m^{th} row of M. Let L be the matrix obtained by deleting first row and n^{th} column of M. Let L^T be the transpose of L. Construct the matrix L^1 from L^T as follows.

Let C_i (for $i=1,2,\ldots,m-1$) denote the columns of L^T. Define $C'_{m-1} = C_{m-1}$ and $C'_i = C'_{i+1} + C_i$, $i = 1,2,\ldots m-2$. Let L^1 be $(C'_1, C'_2,\ldots,C'_{m-1})$. Let r_i (for $i=1,2,\ldots,n-1$) denote the rows of L^1. Define $r'_1 = r_1$ and $r'_i = r'_{i-1} + r_i$ for $i = 2,3,\ldots,n-1$. Let L^* be the matrix whose rows are $r'_1, r'_2, \ldots, r'_{n-1}$. Let $P = U^*L^*$.

Lemma 1:

The sum of the elements on the principal diagonal of P is the number of intersections between the straight line edges in the second quardrant.

Proof:

U^* is an m-1 x n-1 matrix. Here $u_{i,j} = m_{i,j+1}$ for $i = 1,2,\ldots,m-1$ and $j = 2,3,\ldots,n$. Consider $u_{1,1} = m_{1,2}$. Let $m_{1,2} = 1$. Let us denote the corresponding line by $m^{2,1}$. We see that $m^{2,1}$ satisfies the condition 1 with lines $m^{1,i}$ for which the entries $m_{i,1}$ are 1, for $i = 2,3,\ldots,m$. Therefore, $\sum_{i=2}^{m} m_{i,1}$ will give the total number of intersections associated with $m^{2,1}$, by using condition 1.

If we consider the element $u_{1,2} = m_{1,3}$, then we see that the number of intersections associated with the line $m^{3,1}$, by using condition 1, will be given by :

$$\sum_{i=2}^{m} m_{i,1} + \sum_{i=2}^{m} m_{i,2} = \sum_{i=2}^{m} (m_{i,1} + m_{i,2})$$

Thus, for an element $u_{1,s-1} = m_{1,s}$ the number of intersections with the corresponding line $m^{s,1}$, by using condition 1, will be given by :

$$\sum_{i=2}^{m} \sum_{j=1}^{s-1} m_{i,j}$$

for every $s = 2,3,\ldots,n$.

Thus corresponding to the 1st row U_1^* of U^*, we have a column vector t_1^* such that $U_1^* t_1^*$ will give the intersections associated with the elements of U_1^* (by using condition 1). This vector t_1^* is given by:

$$t_1^* = \begin{bmatrix} \sum_{i=2}^{m} \sum_{j=1}^{1} m_{i,j} \\ \sum_{i=2}^{m} \sum_{j=1}^{2} m_{i,j} \\ \vdots \\ \sum_{i=2}^{m} \sum_{j=1}^{n-1} m_{i,j} \end{bmatrix}$$

Consider now any row say U_k^* of U^*. The corresponding column rector t_k^* given by:

$$t_k^* = \begin{bmatrix} \sum_{i=k+1}^{m} \sum_{j=1}^{1} m_{i,j} \\ \sum_{i=k+1}^{m} \sum_{j=1}^{2} m_{i,j} \\ \vdots \\ \sum_{i=k+1}^{m} \sum_{j=1}^{n-1} m_{i,j} \end{bmatrix}$$

and $U_k^* t_k^*$ gives the number of intersections associated with the lines corresponding to the elements of U_k^* (by using condition 1), where $k = 1,2,\ldots\ldots,m-1$.

Now, we form the matrix of the column vectors say $T^* = [\, t_1^*, t_2^*, \ldots\ldots, t_{m-1}^*]$.

Now it is clear that $\sum_{k=1}^{m-1} U_k^* t_k^*$ will give the total number of intersections required. This will be precisely the sum of the elements on the principal diagonal of the product $U^* T^*$.

Now, it is easy to see that $T^* = L^*$

Hence the Lemma is proved.

Now suppose that all the entries of M are 1. This means that, every vertex on the x-axis is joined to every vertex on the y-axis, in the second quadrant. In this case,

$$U^* = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ 1 & \cdot & \cdots & 1 \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{m-1 \times n-1}$$

and

$$L^* = \begin{bmatrix} n-1 & n-2 & \cdots & 2 & 1 \\ 2(n-1) & 2(n-2) & \cdots & 4 & 2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ (m-1)(n-1) & (m-1)(n-2) & \cdots & 2(m-1) & m-1 \end{bmatrix}_{n-1 \times m-1}$$

It is now easily seen that the sum of the elements on the principal diagonal of $P = U^* L^*$ will be given by

$$S_{pd} = \sum_{k=1}^{n-1} k \sum_{s=1}^{m-1} s$$

$$= \frac{mn(m-1)(n-1)}{4}$$

Using this formula, we can verify that the Zarankiewicz's representation realises the minimum number of intersections proved by him, for complete bipartite graphs.

Consider a complete bipartite graph $K_{(p,q)}$. Let m^+ and n^+ denote the vertices on the positive side of the y-axis and positive side of the x-axis, respectively. Let m^- and n^- denote the vertices on the negative side of the y-axis and the negative side of the x-axis, respectively. (There is no edge here between the vertices of the x-axis (y-axis)).

Case 1:

$p = 2k$, $q = 2s$

$m^+ = m^- = k$ and $n^+ = n^- = s$

then the intersection number $I_{p,q}$ is given by:

$$I_{p,q} = 4 \frac{ks\ (k-1)\ (s-1)}{4}$$

$$= (k^2-k)\ (s^2-s)$$

Case 2:

$p = 2k+1$, $q = 2s+1$

$m^+ = k+1$, $m^- = k$ and $n^+ = s+1$, $n^- = s$

$$I_{p,q} = \frac{(k+1)\ (s+1)\ ks}{4} + \frac{(k+1)\ (s)\ (k)\ (s-1)}{4} + \frac{k\,(s+1)(k-1)s}{4} + \frac{ks\ (k-1)\ (s-1)}{4}$$

$$= k^2s^2$$

Case 3:

$p = 2k+1$, $q = 2s$

$m^+ = k+1$, $m^- = k$ and $n^+ = n^- = s$

$$I_{p,q} = \frac{2ks\ (k-1)\ (s-1)}{4} + \frac{2(k+1)\ (s)\ (k)\ (s-1)}{4}$$

$$= k^2\ (s^2-s)$$

Case 4:

$$p = 2k \qquad q = 2s+1$$

$$I_{p,q} = s^2\ (k^2-\ k)$$

This verifies the Zarankiewicz formulae.

Remarks:

The problem that remains to be settled is how the decomposition of the vertices should be done along the x-axis and the y-axis . We suggest the following decomposition. Find a maximal bipartite subgraph (that bipartite subgraph of G, which has at least as many number of edges as any other bipartite subgraph of G). of G. Find out the two disjoint vertex sets joined in this maximal bipartite subgraph. Take one set of vertices along the x-axis and the other set of vertices along the y-axis. The reason for suggesting this decomposition is that by finding a maximal bipartite subgraph of G we are, in some sense, close to the complete bipartite graph.

Another important problem is to estimate the number of different types of intersections. In this x-y axis representation of G, we come across the following different types of intersections between different types of edges of G.

1. The number of intersections taking place between the edges , which are parts of semi-circles, joining the vertices on the x-axis (y-axis) to the vertices on the x-axis (y-axis) .

2. The number of intersections taking place between the edges, which are straight lines, joining the vertices on the x-axis and the vertices on the y-axis.

3. The number of intersections taking place between the straight line edges and the edges, which are parts of the semi-circles.

4. The number of intersections taking place between the edges joining the vertices on the x-axis and the edges joining the vertices on the y-axis.

The first two types of intersections have been estimated (type 1 by Nicholson in (4) and type 2 as discussed in Lemma 1 of Chapter II).

We hope that in estimating the number of intersections of type 3, the concept of the depth of a planar graph (defined in Chapter I) may be useful.

We suggest the following outline of the procedure for minimising the number of intersections in a given graph G.

1. Decide, which vertices of G should lie on the x-axis and the y-axis, by using some decomposition criterian.

2. Estimate the different types of intersections in a given x-y axis representation of G.

3. Let R be an x-y axis representation of G in the plane. Let I_R denote the number of intersections of G in R. Let R' be a representation of G in the plane obtained by interchanging either a pair of vertices on the x-axis or a pair of vertices on the y-axis. Let $I_{R'}$ be the number of intersections of G in R'. Then R is called 2-2 optimal if $I_R \leq I_{R'}$.

Now after the decomposition of the vertices is done, permute the vertices on the x-axis and vertices on the y-axis to get a 2-2 optimal representation of G.

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